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Examples of Landau–Kolmogorov Inequality in Integral Norms on a Finite Interval

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For $r = 2$ and 3, we prove that the equioscillating perfect spline $T_{r,m}$ of degree r on any given finite interval $[a, b]$ is the unique extremal function to the Landau–Kolmogorov problem

$$\|f^{(k)}\|_p \to \sup \ (0 < k < r),$$

over the class of all r times differentiable functions f on $[a, b]$ satisfying the conditions $\|f\|_\infty \leqslant \|T_{r,m}\|_\infty$, $\|f^{(r)}\|_\infty \leqslant \|T^{(r)}_{r,m}\|_\infty$. © 2002 Elsevier Science (USA)

1. INTRODUCTION

In 1912, Hardy and Littlewood [7] found the exact bound for the L_2-norm of the derivative of a smooth integrable function on the real line $\mathbb{R}$ in terms of the L_2-norm of a higher derivative. The problem of estimating the first derivative of f on a given interval I on the basis of the uniform norm of f and f'' on I comes from Landau. He found in [12] the exact upper bound of $\max\{|f'(x)|: x \in I\}$ in the case I is the half-line $\mathbb{R}_+$. A year later Hadamard [6] solved the same problem on the whole line $\mathbb{R}$. Kolmogorov [8] studied the general case of estimating the uniform norm $\|\cdot\|$ of any intermediate derivative of r times differentiable functions on $\mathbb{R}$ and proved the inequality

$$\|f^{(k)}\| \leqslant C_{rk} \|f\|^{(r-k)/r} \, \|f^{(r)}\|^{k/r}$$

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for every function f from the set

$$W_\infty^r(\mathbb{R}) := \{f\colon\ f^{(r-1)} \text{ locally abs. cont. on } \mathbb{R},\ \|f^{(r)}\| < \infty\}.$$

The exact constant C_{rk} was given explicitly and the extremal functions were completely characterized. Schoenberg and Cavaretta [16] described the extremal function to Landau–Kolmogorov problem on the half-line $\mathbb{R}_+$. Motorin [13] did it earlier for $r = 3$ and $k = 1, 2$. However, despite the effort of many mathematicians, the problem is still unsettled in the most interesting case of functions given on a finite interval. Chui and Smith [5] found the solution for $r = 2$ and the case $r = 3$ was studied in [15, 19]. Partial characterization of the extremal function for any r was given by Pinkus [14]. He showed that the extremal function is an almost maximally oscillating perfect spline. In [4] we obtained the exact estimate for the L_p norm on $[-1, 1]$ of any intermediate derivative in the class of oscillating perfect splines of degree r.

Various results on this subject and references can be found in [10, 11, 18]. See also [17] for recent developments.

In this note, we study Landau–Kolmogorov problem in L_p norm on a given finite interval $[a, b]$ for $r = 2$ and 3. Our result is related to a previous publication [3] where we proved a Kolmogorov-type inequality for the L_p-norm of $f^{(k)}$ on a finite interval, if the restrictions on the function and its rth derivative are imposed on the whole real line $\mathbb{R}$. We shall formulate the result precisely since it will be used in the sequel.

Everywhere in this paper, we shall assume that ϕ is a continuously differentiable function on $[0, \infty)$, positive on $(0, \infty)$, and such that $\phi(t)/t$ is non-decreasing. For easy reference we shall denote the class of such functions by Φ. In the literature, they are also called *N-functions* because the integral

$$J_\phi(f; [a, b]) := \int_a^b \phi(|f(x)|)\, dx$$

defines a norm in the space of integrable functions on $[a, b]$. A typical example of N-function is $\phi(x) = x^p$ for $1 \leqslant p < \infty$. Our results are proved for any $\phi \in \Phi$ and thus they hold, in particular, for any L_p-norm. Sometimes, when this will not lead to confusion, we shall omit the specification of the interval $[a, b]$ in the notation of J_ϕ. Also, for simplicity, everywhere we shall use the abbreviated notation $\|\cdot\| := \|\cdot\|_{L_\infty(I)}$ for the uniform norm, if the interval I is clear from the context.

For given $\phi \in \Phi$, positive numbers M_0 and M_r, and a finite interval $[a, b] \subset I$, we define the *Landau–Kolmogorov problem* as

$$J_\phi(f^{(k)}; [a, b]) \to \sup \tag{1}$$

over the set of all functions from $W_\infty^r(I)$ satisfying the conditions

$$\|f\| \leqslant M_0, \qquad \|f^{(r)}\| \leqslant M_r.$$

It was shown in [3] that the so-called *Euler perfect spline*

$$\varphi_r(x) := \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin((2m+1)x - \frac{\pi}{2}r)}{(2m+1)^{r+1}}$$

plays an important role in the study of problem (1) in the case $I = \mathbb{R}$. Recall that an appropriate normalization of Euler's spline is the extremal function in the original Kolmogorov inequality. The function φ_r is a periodic perfect spline of degree r with equidistant knots. Precisely (see, for example, [2]), $\varphi_r^{(r)}(x) = \operatorname{sign} \sin x$. Besides, the following is well known (see [9, Section 2.3]):

$$\varphi_r \text{ is a } 2\pi\text{-periodic function from } W_\infty^r;$$

$$\|\varphi_r\| = K_r$$

where K_r is the famous Favard–Akhiezer–Krein constant,

$$K_r = \frac{4}{\pi} \sum_{v=0}^{\infty} \frac{(-1)^{v(r+1)}}{(2v+1)^{r+1}};$$

$$\|\varphi_r^{(j)}\| = \|\varphi_{r-j}\| = K_{r-j}.$$

Euler spline has only simple zeros and is *equioscillating*, i.e., all local extrema of φ_r are equal in absolute value. For any given positive numbers M_0 and M_r, there exists a unique (up to a shift and multiplication by -1) equioscillating periodic perfect spline of uniform norm M_0 and the norm of its rth derivative equal to M_r (see, for example, [2]). Because of the uniqueness, this spline is given by $c\varphi_r(\gamma(x - \alpha))$ with some constants c and γ determined by M_0 and M_r. In what follows, we shall call this spline *normalized* Euler spline (with respect to M_0 and M_r) and will denote it again by φ_r.

Let $[a, b] \subset \mathbb{R}$. For a given $f \in W_\infty^r(\mathbb{R})$ with

$$M_0(f) := \|f\| < \infty \qquad \text{and} \qquad M_r(f) := \|f^{(r)}\| < \infty,$$

we define by appropriate compression and normalization of φ_r the spline $\varphi_r([a, b], f; x)$ with the properties:

$$\|\varphi_r([a, b], f; \cdot)\| = M_0(f),$$
$$\|\varphi_r^{(r)}([a, b], f; \cdot)\| = M_r(f).$$

Let ω be the half-period of $\varphi_r([a, b], f; x)$ and $b - a = N\omega + 2\theta, 2\theta < \omega$ (N is an integer). We assume in addition that $\varphi_r([a, b], f; x)$ is shifted so that, for a given k (which is clear from the context),

$$\varphi_r^{(k-1)}([a,b],f;a+\theta)=0, \qquad \varphi_r^{(k)}([a,b],f;a+\theta)>0.$$

The following was proved in [3].

THEOREM A. *Assume that $\phi \in \Phi$ and $[a,b]$ is any finite interval on $\mathbb{R}$. Then for every function $f \in W_\infty^r(\mathbb{R})$ and $k=1,\dots,r-1$, we have*

$$\int_a^b \phi(|f^{(k)}(x)|)\,dx \leqslant \int_a^b \phi(|\varphi_r^{(k)}([a,b]),f;x|)\,dx.$$

Moreover, the equality holds only for $f(x)=\pm\varphi_r([a,b],f;x)$, if $b-a \neq m\omega$ for every integer m, and only for the translations of $\varphi_r([a,b],f;x)$, if $b-a=m\omega$ for some integer m.

In the proof of this theorem, we exploited certain extremal properties of the so-called *comparison function*, a notion which goes back to Kolmogorov. Let us recall the definition.

Let $\varphi \in C^1[a,b]$ be a strictly increasing function on $[a,b]$ and $f \in C^1[a_1, b_1]$. We shall say that φ is a comparison function for f (on the intervals $[a,b]$, $[a_1,b_1]$) and shall write f comp φ if:

1. $\varphi(a) \leqslant f(x) \leqslant \varphi(b)$ $\forall x \in [a_1,b_1]$;
2. the equality $f(t)=\varphi(y)$ for some $y \in [a,b]$ and $t \in [a_1,b_1]$ implies

$$|f'(t)| \leqslant |\varphi'(y)|.$$

Each time the comparison is used, the intervals $[a,b]$ and $[a_1,b_1]$ would be specified.

The next proposition summarizes some facts concerning the comparison function, which will be used in the sequel (see for example [3]).

Let us make the convention to denote by $f|_{[c,d]}$ the restriction of the function f on the subinterval $[c,d]$ of its domain.

PROPOSITION 1. (i) *Let $f|_{[a_1,b_1]}$ comp $\varphi|_{[a,b]}$. Then the equalities $f(t_1)=\varphi(y_1)$, $f(t_2)=\varphi(y_2)$ for some $t_1,t_2 \in [a_1,b_1]$ and $y_1,y_2 \in [a,b]$ imply that $|t_2-t_1| \geqslant |y_2-y_1|$.*

(ii) *If $f|_{[a_1,b_1]}$ comp $\varphi|_{[a,b]}$ and f is monotone on $[a,b]$, then for each function $\phi \in \Phi$*

$$\int_{a_1}^{b_1} \phi(|f'(x)|)\,dx \leqslant \int_a^b \phi(|\varphi'(x)|)\,dx.$$

(iii) *Let $\varphi_r(x)$ be a normalized Euler spline with respect to any given positive* *constants M_0 and M_r. Let ω be the half-period of φ_r and $\varphi = \varphi_r|_{[a,a+\omega]}$ where a is such a point that φ is increasing on $[a, a+\omega]$. Then φ is a comparison function for every function $f \in W^r_\infty(\mathbb{R})$ satisfying the conditions $\|f\| \leqslant M_0$, $\|f^{(r)}\| \leqslant M_r$.*

The equality in (i) *and* (ii) *is attained only if the function f is a translation of φ on the corresponding intervals.*

In order to formulate the results of the present paper, we need some definitions. We shall denote by $T_{r,m}([a,b];x)$ the unique perfect spline of degree r with m knots in (a,b) satisfying the conditions

$$\|T_{r,m}([a,b];\cdot)\|_{C[a,b]} = 1;$$

$$T_{r,m}([a,b];x) \text{ equioscillates at } m+r+1 \text{ points in } [a,b].$$

The second condition means that there are $r+m+1$ points $a = t_0 < \cdots < t_{m+r} = b$ such that

$$T_{r,m}([a,b];t_j) = (-1)^{m+r-j}, \qquad j = 0, \dots, m+r.$$

The splines $T_{r,m}$ are called *Tchebycheff perfect splines* since they are extensions of the classical Tchebycheff polynomials

$$T_r(x) := \cos(r \arccos x) \qquad \text{for } -1 \leqslant x \leqslant 1,$$

which, as is well-known, have the *equioscillation property*

$$T_r\left(\cos\frac{j\pi}{n}\right) = (-1)^j, \qquad j = 0, \dots, n.$$

Thus, $T_{r,0}([-1,1];x) \equiv T_r(x)$.

We prove in this paper that for $r = 2$ and 3 the following Landau–Kolmogorov-type inequality holds:

Assume that $I = [a,b]$ is a bounded interval and m is any non-negative integer number. Then

$$J_\phi(f^{(k)}) \leqslant J_\phi(T^{(k)}_{r,m}([a,b];\cdot)), \qquad 1 \leqslant k \leqslant r-1$$

for each function $f \in W^r_\infty([a,b])$ such that

$$\|f\| \leqslant 1, \qquad \|f^{(r)}\| \leqslant \|T^{(r)}_{r,m}([a,b];\cdot)\|.$$

The proof relies on the fact that for $r = 2$ and 3 any function $f \in W^{(r)}_\infty([a,b])$ can be extended from $[a+\lambda, b-\lambda]$ to the whole line $\mathbb{R}$ with preservation of the norms $M_0(f)$ and $M_r(f)$. Then we estimate $\|f^{(k)}\|_p$ on $[a+\lambda, b-\lambda]$ by Theorem A and find, by ad hoc methods, the exact estimate on the end subintervals $[a, a+\lambda]$ and $[b-\lambda, b]$. The number λ depends of the half-period of the corresponding normalized Euler spline.

Remark that the result we formulated above gives a solution to the Landau–Kolmogorov problem for any finite interval $[a, b]$, any $M_0 > 0$ and a special sequence of numbers M_r, namely, for

$$M_r = M_0 \| T_{r,m}^{(r)}([a, b]; \cdot) \|, \qquad m = 0, 1, \ldots .$$

In order to simplify the presentations we shall fix $M_0 = 1$, set $M_r = \|T_r^{(r)}\| = 2^{r-1} r!$, and consider intervals of the special form $[a, b] = I_{2,m} := [-1, 1 + m\sqrt{2}]$, if $r = 2$, and $[a, b] = I_{3,m} := [-1, m+1]$, if $r = 3$, $m = 0, 1, \ldots$. In this case, we shall abbreviate the notation for the corresponding Tchebycheff perfect spline to $T_{r,m}(x)$. It is well known (see [13]) that for $r = 2, 3$ these functions are closely connected with the periodic Euler splines φ_r. Clearly,

$$T_{2,0}|_{[-1,1]} \equiv T_2(x) = 2x^2 - 1 \quad \text{and} \quad T_{3,0}|_{[-1,1]} \equiv T_3(x) = 4x^3 - 3x.$$

The graph of $T_{2,m}$, respectively $T_{3,m}$, consists of shifts of the Tchebycheff polynomial $T_2|_{\left[-\sqrt{2}/2, \sqrt{2}/2\right]}$, respectively $T_3|_{[-1/2, 1/2]}$, with alternating sign (and appropriate continuation of the end pieces).

In Section 2 we show how to continue two and three times differentiable functions beyond their domain and then, on the basis of this result, we prove the Landau–Kolmogorov inequality in Section 3.

2. CONTINUATION OF DIFFERENTIABLE FUNCTIONS

Let I be a given interval (finite or infinite) on the real line $\mathbb{R}$. Introduce the class

$$\Omega^r(I) := \{ f \in W_\infty^r(I) \colon \|f\| \leqslant 1, \|f^{(r)}\| \leqslant T_r^{(r)}(1) \}.$$

The next lemma is about continuation of functions $f \in \Omega^r(I)$ from a given finite interval I to the whole line $\mathbb{R}$ with preservation of the norm.

Lemma 1. *Let $f \in \Omega^2(I)$ with $I := \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$. Then there exist functions $g_1 \in \Omega^2\left(\left(-\infty, \frac{\sqrt{2}}{2}\right]\right)$ and $g_2 \in \Omega^2\left(\left[-\frac{\sqrt{2}}{2}, \infty\right)\right)$ such that*

$$f \equiv g_1 \quad \text{on} \quad \left[0, \frac{\sqrt{2}}{2}\right], \qquad f \equiv g_2 \quad \text{on} \quad \left[-\frac{\sqrt{2}}{2}, 0\right].$$

Proof. Let f be any function from $\Omega^2\left(\left[-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right]\right)$. We shall show how to continue f over $[0,\infty)$. The continuation over $(-\infty,0]$ is similar.

Without loss of generality we may assume that $A := f(0)\geqslant 0$. In view of the lemma conditions $A\leqslant 1$. If $A=1$, then $f'(0)=0$ and evidently we can continue f over $[0,\infty)$ as $f(x)=1$. Thus, we assume below that $0\leqslant A<1$. Let us denote $D := f'(0)$. Consider first the case when $D\geqslant 0$. If $D=0$, then the continuation is trivial—we just set $f(x)=A$ on $[0,\infty)$. Assume next that $D>0$. Then, for every $x\geqslant 0$, we have by Taylor's formula

$$\begin{aligned} f(x) &= A + Dx + \int_0^x (x-t)f''(t)\,dt \\ &\geqslant A + Dx - 4\int_0^x (x-t)\,dt = A + Dx - 2x^2 =: \, p(D;x). \end{aligned}$$

Note that the parabola $p(D;x)$ attains its maximum $M(D)$ at the point $\xi(D) := D/4$ and $M(D) = A + D^2/8$. In particular, for $D = D^* := 4\sqrt{\frac{1-A}{2}}$ we have

$$M(D^*) = 1 \quad \text{and} \quad \xi(D^*) = D^*/4 \leqslant \frac{\sqrt{2}}{2}.$$

We shall prove that

$$D\leqslant D^*.$$

Indeed, assume the contrary. Then

$$f(\xi(D^*))\geqslant p(D;\xi(D^*)) > p(D^*;\xi(D^*)) = M(D^*) = 1$$

and this contradicts the assumption that $f(x)\leqslant 1$ on $\left[0,\frac{\sqrt{2}}{2}\right]$. Therefore $D\leqslant D^*$. But then we can continue f as follows:

$$f(x) = \begin{cases} p(D;x) & \text{for } x\in[0,\xi(D)], \\ M(D) & \text{for } x\geqslant \xi(D). \end{cases}$$

If $D<0$, we conclude as above, on the basis of the assumption $f(x)\leqslant 1$ on $\left[-\frac{\sqrt{2}}{2},0\right]$ that $D\geqslant -D^*$ and

$$-1\leqslant f(x)\leqslant q(D;x) \quad \forall x\in[0,\sqrt{2}/2],$$

where $q(D;x) := A - |D|x + 2x^2$. Denoting this time by $M(D)$ and $\xi(D)$, respectively, the minimum and the point of minimum of $q(D;x)$, we continue

f in the following way:

$$f(x) = \begin{cases} q(D; x) & \text{for } x \in [0, \xi(D)], \\ M(D) & \text{for } x \geqslant \xi(D). \end{cases}$$

The lemma is proved. ∎

Corollary 1. *Let* $f \in \Omega^2([a, b])$ *with any interval* $[a, b]$ *of length* $b - a \geqslant \sqrt{2}$. *Then there exists a function* $g \in \Omega^2(\mathbb{R})$ *such that* $g \equiv f$ on $\left[a + \sqrt{2}/2, b - \sqrt{2}/2\right]$.

The assertion is an immediate consequence of Lemma 1.

Consider next the same problem for $r = 3$. We shall show that functions from $\Omega^3([a, b])$ can be extended from $[a + 1, b - 1]$ to the whole real line with preservation of the norms $\|f\|$ and $\|f^{(3)}\|$. The following observation is crucial for the proof of this claim.

Given the data $\bar{f} := (f(0), f'(0), f''(0))$ we construct the polynomial

$$P(x) := f(0) + f'(0)x + f''(0)x^2/2 + 4\sigma x^3,$$

where $\sigma = 1$, if $f'(0) < 0$, and $\sigma = -1$, if $f'(0) \geqslant 0$. This polynomial has exactly two local extrema and they are situated on the both sides of 0. Let us denote them by $\lambda = \lambda(\bar{f}) \leqslant 0$ and $\mu = \mu(\bar{f}) \geqslant 0$.

We shall say that the data $(f(0), f'(0), f''(0))$ is *extendable* in $\Omega^3(I)$ if there exists a function $f \in \Omega^3(I)$ interpolating these values at 0.

Lemma 2. *The data* $\bar{f}$ *is extendable in* $\Omega^3(\mathbb{R})$ *if and only if* $|P(\lambda)| \leqslant 1$, $|P(\mu)| \leqslant 1$.

Proof. We may assume that $|f'(0)| + |f''(0)| > 0$. Otherwise the assertion is trivial: $f(x) = \text{const.} = f(0)$ is the wanted continuation on $\mathbb{R}$. For the sake of definiteness, let $f'(0) \geqslant 0$ (otherwise we consider the data $-\bar{f}$). Suppose that $g(x)$ is a certain extension of the data $\bar{f}$ in $\Omega^3(\mathbb{R})$. Then $\|g'''\|_{L_\infty(\mathbb{R})} \leqslant 24$ and by Taylor's formula, for $x > 0$, we obtain

$$\begin{aligned} g(x) &= g(0) + g'(0)x + g''(0)x^2/2 + \frac{1}{2}\int_0^x (x - \tau)^2 g'''(\tau)\, d\tau \\ &\geqslant f(0) + f'(0)x + f''(0)x^2/2 + \frac{1}{2}\int_0^x (x - \tau)^2 (-24)\, d\tau \\ &= P(0) + P'(0)x + P''(0)x^2/2 + \frac{1}{2}\int_0^x (x - \tau)^2 P'''(\tau)\, d\tau = P(x). \end{aligned}$$

Therefore, on the interval $[0, \mu]$ (where $P(x)$ is increasing), we have

$$g(x) \geqslant P(x). \tag{2}$$

Similarly, $P(x) \geqslant g(x)$ on $[\lambda, 0]$. Thus,

$$-1 \leqslant g(\lambda) \leqslant P(\lambda) \leqslant P(\mu) \leqslant g(\mu) \leqslant 1$$

and therefore the variation of the polynomial $P(x)$ on $[\lambda, \mu]$ is a lower bound for the variation of any function $g \in \Omega^3(\mathbb{R})$ that interpolates the data $\bar{f}$.

Assume that the data $\bar{f}$ is extendable, that is, there is a function g with the above mentioned properties. Then the last inequality shows that $|P(\lambda)| \leqslant 1$, $|P(\mu)| \leqslant 1$. Conversely, if the data $\bar{f}$ satisfies the conditions in the lemma, then we construct the periodic cubic perfect spline $\tilde{P} \in \Omega^3(\mathbb{R})$, defined on the half-period $[\lambda, \mu]$ as $\tilde{P}(x) = P(x)$, and on the other half $[\mu, 2\mu - \lambda]$ as an even function with respect to μ. Clearly, $\tilde{P}$ is the wanted extension of the data $\bar{f}$. The lemma is proved. ∎

Note that, for extendable $\bar{f}$,

$$\mu(\bar{f}) - \lambda(\bar{f}) \leqslant 1.$$

Indeed, the function $\varphi(x) := \tilde{P}(x) - \tilde{P}\left(\frac{\lambda+\mu}{2}\right)$, where the cubic perfect spline $\tilde{P}$ is associated with the data $\bar{f}$, is an Euler spline situated in a strip with a width less than or equal to 2 and such that $\|\varphi'''\| = 24$. On the other hand, the Euler spline φ_3 normalized to satisfy $\varphi_3(x) = T_{3,m}(x)$ for $x \in \left[-\frac{1}{2}, m + \frac{1}{2}\right]$ (and thus $\|\varphi_3'''\| = 24$, $-1 \leqslant \varphi_3(x) \leqslant 1$ for all $x \in \mathbb{R}$) has a half-period $\omega = 1$. Then $\varphi = c\varphi_3(\gamma(x - \alpha))$, where the constant $c = \|\varphi\|$ and the half-period $\frac{1}{\gamma} = \mu - \lambda$ satisfy the relation

$$c\gamma^3 = 1. \tag{3}$$

More generally, it is true that the half-period of the normalized, with respect to M_0, M_r, Euler spline φ_r is an increasing function on M_0, provided M_r stays fixed. Consequently, for extendable data $\bar{f}$, we have $\|\varphi\| \leqslant 1$ and $\mu - \lambda \leqslant 1$.

As a consequence of (2), Proposition 1 and (3) it is seen that:

The spline $\tilde{P}(x)$ *is the extension of* $\bar{f}$ *in* $\Omega^3(\mathbb{R})$ *with the smallest period.*

Now we are ready to prove our main extension result.

THEOREM 1. *If* $f \in \Omega^3([a, b])$ *and* $b - a \geqslant 2$, *then* f *can be extended from* $[a + 1, b - 1]$ *to the whole real line* $\mathbb{R}$ *as a function from* $\Omega^3(\mathbb{R})$.

Proof. It suffices to prove the assertion only for $[a, b] = [-1, 1]$. In this case, we have to show that the data $\bar{f} := (f(0), f'(0), f''(0))$ is extendable in $\Omega^3(\mathbb{R})$. In order to do this, assume without loss of generality that $f'(0) \geqslant 0$.

Let us consider the polynomial P and the cubic splines $\tilde{P}$ and $\varphi(x) := \tilde{P}(x) - \tilde{P}(\frac{\lambda+\mu}{2})$, associated with $\bar{f}$. According to (2),

$$P(x) \leqslant f(x) \leqslant 1 \text{ on } [0,1]$$

and similarly

$$-1 \leqslant f(x) \leqslant P(x) \text{ on } [-1,0].$$

Thus, in view of Lemma 2, $\tilde{P}(x)$ would be the wanted continuation of f if we show that $[\lambda(\bar{f}), \mu(\bar{f})] \subset [-1,1]$. Assume the contrary, say, $\mu > 1$. It follows from the construction of $P(x)$ that $\lambda(\bar{f}) \leqslant 0$. Therefore, $\mu - \lambda > 1$ and consequently the half-period of $\tilde{P}$ is greater than 1. Thus, because of (3)

$$\frac{\tilde{P}(\mu) - \tilde{P}(\lambda)}{2} = \|\varphi\| > \|\varphi_3\| = 1.$$

We shall apply properties (i) and (iii) of Proposition 1 to the Euler spline φ and the function $f(x) := \varphi_3(x) + C$ with $|C| \leqslant \|\varphi\| - \|\varphi_3\|$. According to (iii), f comp $\varphi|_{[\lambda,\mu]}$. Besides, in view of (i), if t_1, t_2 are such that $f(t_1) = \min_{x\in\mathbb{R}} f(x), f(t_2) = \max_{x\in\mathbb{R}} f(x)$ (for instance $t_1 = 1/2, t_2 = -1/2$ (see the definition of φ_3)), then $|t_2 - t_1| \geqslant (y_2 - y_1)$, where $y_1, y_2 \in [\lambda, \mu] : \varphi(y_1) = -1 + C, \varphi(y_2) = 1 + C$. In other words, every subinterval $[y_1, y_2] \subset [\lambda, \mu]$ on which $\varphi(x)$ takes variation 2 has length $y_2 - y_1 \leqslant 1$. The equality sign cannot be attained because of the strict inequality $\|\varphi_3\| < \|\varphi\|$. This observation implies that $\varphi_3 + C$ cannot be a translation of φ.

In reverse, on every subinterval $[\alpha, \alpha+1] \subset [\lambda, \mu]$ the function $\varphi(x)$, and consequently $P(x)$, has a variation > 2. In particular $P(1) - P(0) > 2$, which leads to $P(1) > 1$ or $P(0) < -1$, a contradiction. The proof is complete. ∎

Remark. Note that Theorem 1 is no longer true for intervals $[a,b]$ of length $b - a < 2$ and a cut of length < 1.

3. EXACT ESTIMATES FOR THE DERIVATIVES

In this section we give a Landau–Kolmogorov-type inequalities for the derivatives. Some of the reasonings we are going to apply in the proof are of independent interest. That is why we give them separately as auxiliary lemmas.

With any $f \in C[a,b]$ we associate its *non-increasing rearrangement* $r(f;t)$ defined on $[0, b-a]$ by

$$r(f;t) := \inf\{y : m(f;y) \leqslant t, \ t \in [0, b-a]\},$$

where

$$m(f;y) := \mathrm{mes}\{t : t \in [a,b], f(t) > y\}.$$

For each pair f, g of integrable and non-negative functions on $[a,b]$, we shall write $f \prec g$ to denote that

$$\int_0^t r(f;x)\,dx \leqslant \int_0^t r(g;x)\,dx \quad \forall t \in [0, b-a].$$

The proof of the following theorem can be found in [10].

THEOREM B. *If the integrable functions $f(x)$ and $g(x)$ on $[a,b]$ satisfy $|f| \prec |g|$, then for any N-function $\phi \in \Phi$ we have*

$$\int_a^b \phi(|f(x)|)\,dx \leqslant \int_a^b \phi(|g(x)|)\,dx.$$

Conversely, if the above relation holds for each N-functions, then $|f| \prec |g|$.

LEMMA 3. *For every continuous function f on $[a,b]$, we have*

$$\omega(r(f);\delta) \leqslant \omega(f;\delta).$$

Proof. For any fixed function $f \in C[a,b]$ and a number δ, $0 < \delta \leqslant b-a$, take any points $x < y$ in $[0, b-a]$ such that $|x-y| \leqslant \delta$. Without loss of generality we may assume that $f(x) \geqslant 0$ on $[a,b]$ (otherwise we consider $f(x) + C, C = \text{const.} > 0$).

Let ξ, η be points from $[a,b]$ for which

$$r(f;x) = f(\xi), \qquad r(f;y) = f(\eta).$$

Assume, for the sake of definiteness, that $\xi < \eta$. Let us choose the point

$$\xi_0 := \sup\{t : t \in [\xi, \eta], f(t) = f(\xi)\}$$

and then choose

$$\eta_0 := \inf\{t : t \in [\xi_0, \eta], f(t) = f(\eta)\}.$$

Clearly,

$$f(\eta_0) \leqslant f(t) \leqslant f(\xi_0) \quad \text{for } t \in [\eta_0, \xi_0].$$

Therefore,

$$\begin{aligned} y - x &= \operatorname{mes}\{t : t \in [a,b], r(f;y) \leqslant f(t) \leqslant r(f;x)\} \\ &\geqslant \operatorname{mes}\{t : t \in [\xi_0,\eta_0], r(f;y) \leqslant f(t) \leqslant r(f;x)\} \\ &= \operatorname{mes}\{t : t \in [\xi_0,\eta_0], f(\eta_0) \leqslant f(t) \leqslant f(\xi_0)\} = |\xi_0 - \eta_0|. \end{aligned}$$

Then

$$\begin{aligned} |r(f;x) - r(f;y)| = |f(\xi_0) - f(\eta_0)| &\leqslant \omega(f;|\xi_0 - \eta_0|) \\ &\leqslant \omega(f;|x-y|) \leqslant \omega(f;\delta) \end{aligned}$$

and we conclude that

$$\omega(r(f);\delta) \leqslant \omega(f;\delta).$$

The proof is complete. ∎

With any interval $[a,b] \subset \mathbb{R}$ we associate the parabola $\tau(x) = \tau([a,b];x)$ defined uniquely by the conditions:

$$\tau(a) = -1, \qquad \tau(b) = 1, \qquad \tau''(x) = 4.$$

LEMMA 4. *Let $\phi \in \Phi$ and $f \in \Omega^2([a,b])$ where $0 < \delta := b - a \leqslant 1$. Then we have*

$$\int_a^b \phi(|f'(x)|)\, dx \leqslant \int_a^b \phi(|\tau'([a,b];x|)\, dx. \tag{4}$$

The equality is attained only for $f = \tau$ (up to symmetry), provided $\phi(x)/x$ is strictly increasing.

Proof. Assume for simplicity that $a = 0$. Let $f \in \Omega^2([0,\delta])$. If $f'(x_0) = 0$ for some point $x_0 \in [0,\delta]$, then clearly

$$|f'(x)| \leqslant 4|x - x_0| \leqslant \tau'(x) \quad \text{for } x \in [x_0,\delta],$$

$$|f'(x)| \leqslant 4|x - x_0| \leqslant \tau'(x_0 - x) \quad \text{for } x \in [0,x_0]$$

and thus inequality (4) is true. Assume now that $f'(x) \neq 0$ on $[0,\delta]$. Then f is a monotone function and consequently its total variation on $[0,\delta]$ is bounded by 2, which is the variation of τ. In other words,

$$\int_0^\delta |f'(x)|\, dx \leqslant \int_0^\delta |\tau'(x)|\, dx. \tag{5}$$

Consider the function f'. It does not change sign on $[0,\delta]$. Assume for definiteness that $f'(x)\geqslant 0$. Since $f\in\Omega^2([0,\delta])$, we have

$$|f'(x)-f'(y)|\leqslant 4|x-y| \;\; \forall x,y\in[0,\delta].$$

By Lemma 3, the same inequality holds for the non-increasing rearrangement $r(f';x)$ of f'. Since $|r(\tau';x)-r(\tau';y)|=4|x-y|$, we then obtain

$$|r(f';x)-r(f';y)|\leqslant|r(\tau';x)-r(\tau';y)| \;\; \forall x,y\in[0,\delta].$$

This means that the slope of $r(f')$ is less than or equal to the slope of $r(\tau')$ at every point $x\in[0,\delta]$. Thus the graph of $r(f')$ cannot intersect the graph of $r(\tau')$ more than once. As a consequence of this observation we conclude that:

(i) $r(f';x)\leqslant r(\tau';x)$ $\forall x\in[0,\delta]$, or

(ii) the graph of $r(f')$ intersects the graph of $r(\tau')$ at a certain point $y\in(0,\delta)$ and

$$r(f';x)\leqslant r(\tau';x) \quad \text{for } 0\leqslant x\leqslant y, \quad r(f';x)\geqslant r(\tau';x) \quad \text{for } y\leqslant x\leqslant\delta.$$

In both cases we have

$$r(f';0)\leqslant r(\tau';0). \tag{6}$$

Note that the situation $r(f';x)\geqslant r(\tau';x)$ for all $x\in[0,\delta]$ cannot occur because of (5). Also, (ii) with the inverse inequalities cannot occur since the slope of $r(f')$ would be bigger than the slope of $r(\tau')$ at the point of intersection.

The observations above yield

$$\int_0^t r(f';x)\,dx\leqslant\int_0^t r(\tau';x)\,dx \quad \forall t\in[0,\delta]. \tag{7}$$

Indeed, in case (i) (7) is obvious. In case (ii), in view of (6) inequality (7) holds for $0\leqslant t\leqslant y$. If we assume now that (7) does not hold for some $t>y$, then it would not hold also for any other $t\in[y,\delta]$ and, in particular, for $t=\delta$. But this contradicts (5). Therefore (7) is true. Let us note in this place that when $\phi(x)/x$ is strictly increasing, the equality in (7) is a necessary condition for the extremality of f. If the equality sign holds in (7) for each t, then $r(f')\equiv r(\tau')\equiv\tau'(\delta-\cdot)$. The last identity follows from the fact that $\tau'(x)$ is a monotone function on $[0,\delta]$. But $\tau'(\delta-x)=4(\delta-x)$ and thus for each extremal function f we have $r(f';x)=4(\delta-x)$. This implies that $f'(x)=4(\delta-x)$ (up to symmetry) and consequently $f=\tau$, up to symmetry.

An application of Theorem B completes the proof of the lemma.

Remark. It is worth mentioning that an analog of Lemma 4 holds also for intervals of length $\delta, 1<\delta\leqslant 2$, with a majorating function $T(x) := T_2(x-1-a)$ instead of τ.

Indeed, let $f \in \Omega^2([a,b])$. If f is a monotone function on $[a,b]$, then the inequality $J_\phi(f';[a,b])\leqslant J_\phi(T';[a,a+1])$ follows as in the proof of Lemma 4. Therefore, we can assume that f'vanishes at some point $x_0 \in [a,b]$.

If $x_0 \in [a+1,b]$, then according to Lemma 4, $|f'| \prec |T'|$ on $[a,a+1]$, while, on $[a+1,b], r(|f'|)$ is majorized by $r(|T'|)$. The case $x_0 \in [a,b-1]$ is a symmetric to the above one. So, it remains to consider the situation when $x_0 \in (b-1,a+1)$. In such a case our claim follows immediately from the relations $|f'| \prec 4|x-x_0| \prec |T'|$. The first one is obvious, while the second can be verified constructing $r(4|x-x_0|)$ and $r(|T'|)$ explicitely, or comparing the parabolas $2(x-x_0)^2-1$ and $T(x)$.

THEROEM 2. *Let $\phi \in \Phi$. For any fixed non-negative integer m and every function $f \in \Omega^2(I_{2,m})$ we have*

$$J_\phi(f')\leqslant J_\phi(T'_{2,m}).$$

The equality is attained only for $f = \pm T_{2,m}$.

Proof. In case $m=0$ the spline $T_{2,0}$ coincides with the Tchebycheff polynomial $T_2(x)=2x^2-1$ (if we stipulate $I_{2,0}=[-1,1]$) and the theorem follows from Lemma 4, applied to both subintervals $[-1,0]$ and $[0,1]$.

Assume now that $m>0$. Let $[a,b] := I_{2,m}$. We partition the interval $[a,b]$ into three parts: $[a,b]=[a,a+1]\cup[a+1,b-1]\cup[b-1,b]$. By Corollary 1, there exists a function $g \in \Omega^2(\mathbb{R})$ which coincides with f on $[a+1,b-1]$. Then, by Theorem A,

$$\int_{a+1}^{b-1} \phi(|f'(x)|)\,dx \leqslant \int_{a+1}^{b-1} \phi(|\varphi_2'([a,b],f;x)|)\,dx = \int_{a+1}^{b-1} \phi(|T'_{2,m}(x)|)\,dx$$

with φ_2 defined by $M_0=1$ and $M_2=4$. Applying Lemma 4 we estimate $J_\phi(f')$ on $[a,a+1]$ and on $[b-1,b]$ by $J_\phi(\tau'([a,a+1];\cdot))$ and $J_\phi(\tau'([b-1,b];\cdot))$, respectively. Finally, observing that $\tau([a,a+1];\cdot)$ and $\tau([b-1,b];\cdot)$ are the restrictions of $T_{2,m}$ on $[a,a+1],[b-1,b]$, respectively, we complete the proof. Note that problem (1) for $r=2$ and interval $[a,b]$ of arbitrary length $\delta>2$ is studied in [13a]. ∎

Now we turn to the case $r=3$. First we show an extremal property of the Tchebycheff polynomial T_3 in the class $\Omega^3([-1,1])$. To this purpose, let us mention the following observation.

LEMMA 5. *Let $f \in \Omega^3([a,b])$ with $b-a \geqslant 2$. Assume that $f'(\zeta)=0$ for some point $\zeta \in [a,b]$. Then*

$$|f''(\zeta)| \leqslant 12.$$

Proof. Suppose, for the sake of definiteness, that $\zeta \leqslant (a+b)/2$ and $f''(\zeta) \geqslant 0$. Note that under this stipulation $[\zeta, \zeta+1] \subset [a,b]$. If we assume now that $f''(\zeta) > 12$, then

$$f''(x) > 12 - 24(x-\zeta) \;\; \forall x > \zeta,$$

$$f'(\zeta + x) = \int_{\zeta}^{\zeta+x} f''(t)\,dt > \int_0^x 12 - 24t\,dt = 12x(1-x),$$

$$f'(\zeta + x) > 12x(1-x), \;\; \forall x \in [0,1] \Rightarrow \int_0^1 f'(\zeta + x)\,dx > 2.$$

The last inequality implies $(f(\zeta+1) - f(\zeta)) > 2$ which contradicts the assumption that f is bounded by 1. The proof is complete. ∎

Now we are prepared to prove the extremal property of T_3.

THEOREM 3. *Let $\phi \in \Phi$. For each $f \in \Omega^3([-1,1])$, we have*

$$\int_0^1 \phi(|f^{(k)}(x)|)\,dx \leqslant \int_0^1 \phi(|T_3^{(k)}(x)|)\,dx, \quad k = 1,2. \tag{8}$$

The equality holds only for $f = \pm T_3$.

Proof. The proof is different for $k=1$ and 2. We start with $k=1$. Consider first the case when f is monotone on [0, 1]. Assume for definiteness that f is increasing. We shall show that $T_3|_{[1/2,1]}$ is a comparison function for $f|_{[0,1]}$. Indeed, assume the contrary. Then, for some points $t \in [0,1]$, $y \in [1/2, 1]$ such that $f(t) = T_3(y)$, we have

$$f'(t) > T_3'(y). \tag{9}$$

According to Theorem 1 there exists a function $g \in \Omega^3((-\infty, 1])$ such that $g \equiv f$ on $[0,1]$. We shall work with g instead of f. Consider now the difference $h(x) := g(x-(y-t)) - T_3(x)$. Because of (9), which holds for the function g too, $h(x)$ will have at least 2 zeros in $[1/2, 1]$ and 2 other zeros coming from the intersection of $g(x-(y-t))$ and each of the other two "harmonics" (monotone pieces of the graph) of $T_3(x)$. Thus, h will have at least 4 zeros, of which one is simple (and consecutively, it is a change of sign for $h(x)$). By Rolle's theorem $h^{(3)}$ will have then at least one sign change. But

$\operatorname{sign} h^{(3)}(x) = \operatorname{sign} T_3^{(3)}(x) = 1$, a contradiction. Therefore, (9) does not hold. Then an application of Proposition 1 yields estimate (8) for $k = 1$.

Next, we consider the case when f has at least one extremum in $(0, 1)$. Denote by ζ the largest point such that $f'(\zeta) = 0$. Then f is monotone, say increasing, on $[\zeta, 1]$. By Theorem 1, there exists a continuation g of f on $(-\infty, 1]$. Assume first that $\zeta \in (0, 1/2]$. Then we cut g at the point ζ and redefine it on $[\zeta, \infty)$ as an even function with respect to ζ. Clearly, the new function g belongs to $\Omega^3(\mathbb{R})$ and coincides with f on $[0, \zeta]$. Besides, because of the symmetry with respect to ζ, we have

$$\int_0^{\zeta} \phi(|g'(x)|)\, dx = \int_{\zeta}^{2\zeta} \phi(|g'(x)|)\, dx.$$

On the other hand, by Theorem A,

$$\int_0^{2\zeta} \phi(|g'(x)|)\, dx \leqslant 2 \int_0^{\zeta} \phi(|T_3'(x)|)\, dx,$$

because the corresponding Euler spline $\varphi_3([0, 2\zeta], g; x)$ is also symmetric with respect to ζ. Precisely $\varphi_3([0, 2\zeta], g; x) = -T_3(x - \zeta)$ on $[0, 2\zeta]$. Therefore,

$$\int_0^{\zeta} \phi(|f'(x)|)\, dx \leqslant \int_0^{\zeta} \phi(|T_3'(x)|)\, dx. \tag{10}$$

Since $f|_{[\zeta,1]}$ comp $T_3|_{[1/2,1]}$ and f is increasing on $[\zeta, 1]$, we have

$$J_{\phi}(f'; [\zeta, 1]) \leqslant J_{\phi}(T_3'; [1/2, 1]).$$

Consequently,

$$\begin{aligned} \int_0^1 \phi(|f'(x)|)\, dx &\leqslant \int_0^{\zeta} \phi(|T_3'(x)|)\, dx + \int_{1/2}^1 \phi(|T_3'(x)|)\, dx \\ &\leqslant \int_0^1 \phi(|T_3'(x)|)\, dx \end{aligned}$$

and (8) is proved also in this case.

Let $\zeta > 1/2$. Then we construct again the function g as in the previous case. A careful application of Theorem A yields this time, in place of (10), the following

$$\int_0^{\zeta} \phi(|f'(x)|)\, dx \leqslant \int_{1/2-\zeta}^{1/2} \phi(|T_3'(x)|)\, dx. \tag{11}$$

Set $s_1 := \zeta - 1/2$, $s_2 := 1 - \zeta$. In view of Proposition 1 and the simple fact that $T_3'|_{[1/2,1]}$ is an increasing function, it follows that

$$\int_{1/2-\zeta}^{0} \phi(|T_3'(x)|)\,dx < \int_{y_1}^{y_2} \phi(|T_3'(x)|)\,dx \leqslant \int_{1-\delta}^{1} \phi(|T_3'(x)|)\,dx,$$

where $y_1, y_2 \in [1/2, 1]$ are such that $T_3(y_1) = 0$, $T_3(y_2) = T_3(1/2 - \zeta)$, and $\delta := y_2 - y_1$. Moreover, $\delta < s_1$ and consequently

$$\int_{1/2-\zeta}^{0} \phi(|T_3'(x)|)\,dx < \int_{1-s_1}^{1} \phi(|T_3'(x)|)\,dx. \tag{12}$$

On the other hand, by Lemma 5,

$$|f''(\zeta)| \leqslant 12 = T_3''(1/2).$$

In addition

$$f'(\zeta) = T_3'(1/2) = 0 \quad \text{and} \quad |f'''(x)| \leqslant 24 = T_3'''(x).$$

Then, by Taylor's formula, for $t > 0$,

$$|f'(\zeta + t)| \leqslant T_3'(1/2 + t)$$

and thus

$$\int_{\zeta}^{1} \phi(|f'(x)|)\,dx \leqslant \int_{1/2}^{1/2+s_2} \phi(|T_3'(x)|)\,dx. \tag{13}$$

Applying the inequalities (11)–(13) and taking into account that $s_1 + s_2 = 1/2$, we obtain

$$\begin{aligned} J_\phi(f';[0,1]) &= J_\phi(f';[0,\zeta]) + J_\phi(f';[\zeta,1]) \\ &\leqslant J_\phi(T_3';[1/2-\zeta,1/2]) + J_\phi(T_3';[1/2,1/2+s_2]) \\ &< J_\phi(T_3';[1-s_1,1]) + J_\phi(T_3';[0,1/2]) + J_\phi(T_3';[1/2,1-s_1]) \\ &= J_\phi(T_3';[0,1]). \end{aligned}$$

Inequality (8) is proved for $k = 1$. In view of the statements concerning the equality cases in Proposition 1 and Theorem A we see that the equality in (8) is attained only for $\zeta = 1/2$ and $f = \pm T_3$.

Consider now the case $k = 2$. Let us note that

$$|f'(0)| \leqslant |T_3'(0)| = 3, \quad |f'(1)| \leqslant |T_3'(1)| = 9 \qquad \forall f \in \Omega^3([-1,1]). \tag{14}$$

The first inequality holds because the data $\bar{f} = (f(0), f'(0), f''(0))$ is extendable in $\Omega^3(\mathbb{R})$. The second follows from the original L_∞ Kolmogorov problem for finite intervals (see [15] or [19]). Both inequalities can be derived directly by the reasoning we used at the beginning of this proof.

It follows from estimates (14) that if $m_1 := \min f'(x)$ and $m_2 := \max f'(x)$ on $[0,1]$, then $m_2 - m_1 \leqslant 12$. The claim is clear if $f'(x)$ is monotone. If $f''(x)$ vanishes at a certain point $\xi \in (0,1)$, then integrating the inequality $|f''(x)| \leqslant 24|x-\xi|$ we conclude that even the variation $\bigvee(f')$ on $[0,1]$ cannot exceed 12 (the last value being reached only for T_3'). Then by Lemma 4, applied to the function $\frac{1}{6}\left(f' - \frac{m_2+m_1}{2}\right)$ on the interval $[0,1]$, we obtain

$$\int_0^1 \phi_1(|f''(x)|/6)\,dx \leqslant \int_0^1 \phi_1(|T_3''(x)|/6)\,dx,$$

where $\phi_1(x) = \phi(6x)$ is evidently from Φ.

The equality sign holds above only if $f' = \pm T_3' + \text{const.}$, which, in view of (14), implies $f' = \pm T_3'$. The latter yields $f = \pm T_3 + \text{const.}$ on $[0,1]$, and consequently $f = \pm T_3$. The proof is complete. ∎

THEOREM 4. *Assume that* $f \in \Omega^3([a,b])$ *with* $b - a = 2 + m$. *Then*

$$J_\phi(f^{(k)}) \leqslant J_\phi\left(T_{3,m}^{(k)}\right), \qquad k = 1, 2$$

and the equality is attained only for $f \equiv \pm T_{3,m}(\cdot)$.

Proof. As in the proof of Theorem 2 we consider f on $[a, a+1], [a+1, b-1], [b-1, b]$ and apply Theorems A and 3 to estimate J_ϕ on these subintervals. ∎

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